skin-friction coefficient increase with the increasing values of m as in the case of constant properties for all x^*.

The results indicate that the variations of thermal conductivity and viscosity to the local Nusselt number and the local skin-friction coefficient are quite significant. Care must be taken in using the constant properties solution.

The variational method combined with CSMP program has been found to be fruitful in the solution of this complex problem. Particularly, it was found that the CSMP program in solving this type of problem is very simple and straightforward. The great potential of such a combined technique and the CSMP program are still to be fully exploited in other engineering problems.

References

[1] Kays, W. M., *Convective Heat and Mass Transfer*, McGraw-Hill, New York, 1966, Chaps. 7, 10 and 12.

[2] Kumar, A., "Application of a CSMP Technique and Variational Method to Solve External Flow Problems with Variable Physical Properties," M.A.Sc. thesis, 1972, Univ. of Ottawa, Canada.

[3] Glansdorff, P. and Prigogine, I., "On a General Evolution Criterion in Macroscopic Physics," *Physica* 30, 1964, pp. 351–374.

[4] Hays, D. F., "An Extended Variational Method Applied to Poiseuille Flow: Temperature Dependent Viscosity," *International Journal of Heat and Mass Transfer*, Vol. 9, 1966, pp. 165–170.

[5] Donnely, R. J., Herman, R., and Prigogine, I., *Symposium on Non-equilibrium Thermodynamics: Variational Techniques and Stability*, Pt. I, University of Chicago Press, 1966.

[6] Schechter, R. S., *The Variational Method in Engineering*, McGraw-Hill, New York, 1967, Chap. 5.

[7] Su, Y. L., "Application of a Variational Method to Flow over a Flat Plate in the Entrance Region with Variable Physical Properties," M.A.Sc. thesis, 1970, Univ. of Ottawa, Canada; also *International Journal of Heat and Mass Transfer*, to be published.

[8] Falkner, V. M. and S. W. Skan, "Some Approximate Solutions of the Boundary-Layer Equations," *Philosophical Magazine*, Vol. 12, No. 7, 1931, pp. 865-896.

[9] Eckert, E. R. G., "Die Berechnung des Warmeubergangs in der laminaren Grenzschicht," *VDI-Forschungsh*, Vol. 416, 1942, pp. 1–24.

[10] Yang, K. T., "Laminar Forced Convection of Liquids in Tubes with Variable Viscosity," *Transactions of the ASME, Ser. C: Journal of Heat Transfer*, Vol. 84, 1962, pp. 353–362.

[11] Schade, K. W. and D. M. McEligot, "Cartesian Graetz Problems with Air Property Variation," *International Journal of Heat and Mass Transfer*, Vol. 14, 1971, pp. 553–666.

A Correction to "Lifting-Line Theory as a Singular Perturbation Problem"

KEITH P. KERNEY*
Naval Ship Research and Development Center, Bethesda, Md.

THE purpose of this Note is to correct the formula for the lift-curve slope of a high-aspect-ratio elliptical wing in incompressible inviscid uniform flow given by Van Dyke in Refs. 1 and 2. Equation (2.29) of Ref. 1 states that the lift-curve slope is given by

$$\frac{dC_L}{d\alpha} = \pi \int_{-1}^{1} h(s)\frac{\Gamma}{\Gamma_\infty}\,ds \qquad (1)$$

where s is the spanwise Cartesian coordinate made dimensionless by division by the semispan, $h(s)$ is the local half-chord made dimensionless by division by the semispan divided by the aspect ratio, and Γ/Γ_∞ is given as a function of $h(s)$ and A, the aspect ratio, by Eq. (2.27) of Ref. 1 as

$$\frac{\Gamma}{\Gamma_\infty} = 1 - \frac{1}{2A}\oint_{-1}^{1}\frac{dh(\sigma)}{d\sigma}\frac{d\sigma}{s-\sigma} + \frac{\log A}{4A^2}\left[2\left(\frac{dh}{ds}\right)^2 + 3h\frac{d^2h}{ds^2}\right] + \frac{1}{4A^2}\left[\left(2\log\frac{4}{h}-\frac{3}{2}\right)\left(\frac{dh}{ds}\right)^2 + \left(3\log\frac{4}{h}+\frac{1}{2}\right)h\frac{d^2h}{ds^2} - 2\frac{d}{ds}\oint_{-1}^{1}\frac{h_2(\sigma)}{s-\sigma}\,d\sigma + \frac{h}{2}\frac{d^3}{ds^3}\int_{-1}^{1}h(\sigma)\,\mathrm{sgn}(s-\sigma)\log|s-\sigma|\,d\sigma + \frac{1}{2}\frac{d^3}{ds^3}\int_{-1}^{1}h^2(\sigma)\,\mathrm{sgn}(s-\sigma)\log|s-\sigma|\,d\sigma\right] + o(A^{-2}) \qquad (2)$$

which is an asymptotic expansion valid as $A \to \infty$. In Eq. (2.18) of Ref. 1 $h_2(s)$ is defined as

$$h_2(s) = -\frac{h(s)}{2}\oint_{-1}^{1}\frac{dh(\sigma)}{d\sigma}\frac{d\sigma}{s-\sigma} \qquad (3)$$

Reference 1 states that the first derivatives of the last two integrals in Eq. (2) can be evaluated by the finite-part method.

The wing with elliptic planform is described by

$$h(s) = (4/\pi)(1-s^2)^{1/2} \qquad (4)$$

Then, with use of Eqs. (3) and (4) and Eqs. (7) and (14) of Appendix B of Ref. 3, Eq. (2) reduces to

$$\frac{\Gamma}{\Gamma_\infty} = 1 - \frac{2}{A} - \frac{4}{\pi^2}\frac{\log A}{A^2}\frac{3-2s^2}{1-s^2} + \frac{4}{\pi^2}\frac{1}{A^2}\left[\pi^2 + \frac{3}{2} - \frac{2}{1-s^2} - \frac{3-2s^2}{1-s^2}\log\frac{\pi}{(1-s^2)^{1/2}} + \frac{(1-s^2)^{1/2}}{2}\frac{d^3}{ds^3}\int_{-1}^{1}(1-\sigma^2)^{1/2}\,\mathrm{sgn}(s-\sigma)\log|s-\sigma|\,d\sigma + \frac{1}{2}\frac{d^3}{ds^3}\int_{-1}^{1}(1-\sigma^2)\,\mathrm{sgn}(s-\sigma)\log|s-\sigma|\,d\sigma\right] + o(A^{-2}) \qquad (5)$$

The last integral in Eq. (5) can be evaluated easily. Taking its third derivative gives

$$\frac{1}{2}\frac{d^3}{ds^3}\int_{-1}^{1}(1-\sigma^2)\,\mathrm{sgn}(s-\sigma)\log|s-\sigma|\,d\sigma = -\log(1-s^2) + \frac{2}{1-s^2} \qquad (6)$$

The first derivative of the remaining integral is evaluated by use of the finite-part method. Following Heaslet and Lomax[4]:

$$\frac{d}{ds}\int_{-1}^{1}(1-\sigma^2)^{1/2}\,\mathrm{sgn}(s-\sigma)\log|s-\sigma|\,d\sigma = \frac{d}{ds}\left\{\int_{-1}^{s}[(1-\sigma^2)^{1/2}-(1-s^2)^{1/2}]\log(s-\sigma)\,d\sigma + (1-s^2)^{1/2}\int_{-1}^{s}\log(s-\sigma)\,d\sigma - \int_{s}^{1}[(1-\sigma^2)^{1/2}-(1-s^2)^{1/2}]\log(\sigma-s)\,d\sigma - (1-s^2)^{1/2}\int_{s}^{1}\log(\sigma-s)\,d\sigma\right\} =$$

$$\int_{-1}^{s}\frac{(1-\sigma^2)^{1/2}-(1-s^2)^{1/2}}{s-\sigma}\,d\sigma + (1-s^2)^{1/2}\log(1+s) + \int_{s}^{1}\frac{(1-\sigma^2)^{1/2}-(1-s^2)^{1/2}}{\sigma-s}\,d\sigma + (1-s^2)^{1/2}\log(1-s) =$$

$$-(1-s^2)^{1/2} + s\sin^{-1}s + (1-s^2)^{1/2}\log[2(1-s^2)] + s\pi/2 - s\pi/2 - (1-s^2)^{1/2} + s\sin^{-1}s + (1-s^2)^{1/2}\log[2(1-s^2)]$$

Therefore,

$$\frac{1}{2}\frac{d^3}{ds^3}\int_{-1}(1-\sigma^2)^{1/2}\,\mathrm{sgn}(s-\sigma)\log|s-\sigma|\,d\sigma = \frac{d^2}{ds^2}\{(\zeta) - (1-s^2)^{1/2} + s\sin^{-1}s + (1-s^2)^{1/2}\log[2(1-s^2)]\}$$

Received March 16, 1972; revision received May 24, 1972. This research was supported by the Naval Ship Systems Command under task J-15090. The author gratefully acknowledges the assistance which M. Van Dyke provided in a series of letters.

Index categories: Airplane and Component Aerodynamics; Subsonic and Transonic Flow.

* Aerospace Engineer. Member AIAA.

$$= \frac{1}{(1-s^2)^{1/2}} + \frac{2s^2}{(1-s^2)^{3/2}} - \frac{\log[2(1-s^2)]}{(1-s^2)^{3/2}} \quad (7)$$

Substitution of Eqs. (6) and (7) into Eq. (5) results in

$$\frac{\Gamma}{\Gamma_\infty} = 1 - \frac{2}{A} - \frac{4}{\pi^2}\frac{\log A}{A^2}\frac{3-2s^2}{1-s^2} + \frac{4}{\pi^2}\frac{1}{A^2}\left\{\pi^2 + \frac{5}{2} - \frac{3-2s^2}{1-s^2}\right.$$
$$\left.\log\frac{\pi}{(1-s^2)^{1/2}} - \log(1-s^2) + \frac{2s^2}{1-s^2} - \frac{\log[2(1-s^2)]}{1-s^2}\right\} + o(A^{-2}) \quad (8)$$

which differs from Eq. (3.10) of Ref. 1 in the coefficient of A^{-2}. Finally, the lift-curve slope is found using Eqs. (1, 4, and 8) and Eqs. (864.31) and (864.32) of Ref. 5 as

$$\frac{dC_L}{d\alpha} = 2\pi\left[1 - \frac{2}{A} - \frac{16}{\pi^2}\frac{\log A}{A^2} + \frac{4}{\pi^2}\frac{1}{A^2}\left(\frac{9}{2} + \pi^2 - 4\log\pi\right)\right] + o(A^{-2}) \quad (9)$$

Equation (9) differs from Eq. (3.11) of Ref. 1 in the presence of 9/2 in place of 7/2. If the series is recast as a fraction the 7 in Eq. (3.13) of Ref. 1, Eq. (10.27) of Ref. 2, and Eq. (12) of Ref. 6 should be replaced by 9.

References

[1] Van Dyke, M., "Lifting-Line Theory as a Singular Perturbation Problem," *Archiwum Mechaniki Stosowanej*, Vol. 16, No. 3, 1964, pp. 601–614; also Rept. 165, Aug. 1963, Dept. of Aeronautics and Astronautics, Stanford Univ.; also *Fluid Dynamics Transactions*, Vol. 2, Pergamon Press, Oxford, 1965, pp. 369–382; also *Journal of Applied Mathematics and Mechanics*, Vol. 28, No. 1, Oct. 1964, pp. 90–102.

[2] Van Dyke, M., *Perturbation Methods in Fluid Mechanics*, Academic Press, New York, 1964, Secs. 9.2–9.5.

[3] Van Dyke, M., "Second-Order Subsonic Airfoil Theory including Edge Effects," Rept. 1274, 1956, NACA.

[4] Heaslet, M. A. and Lomax, H., "Supersonic and Transonic Small-Perturbation Theory," *General Theory of High-Speed Aerodynamics*, edited by W. R. Sears, Princeton University Press, Princeton, N. J., 1954, Sec. 14.1.

[5] Dwight, H. B., *Tables of Integrals and Other Mathematical Data*, 4th ed., MacMillan, New York, 1961.

[6] Germain, P., "Recent Evolution in Problems and Methods in Aerodynamics," *Journal of the Royal Aeronautical Society*, Vol. 71, No. 682, Oct. 1967, pp. 673–691.

Prandtl Eddy Viscosity Model for Coaxial Jets

HERMANN VIETS*
Aerospace Research Laboratories/LE, Wright-Patterson Air Force Base, Ohio

A RECENT paper by Harsha[1] presents a rather complete comparison with experimental data of freejet calculations employing eddy viscosity models. One of the conclusions is that of all the theoretical models depending solely on the local mean flow properties, the Prandtl[2] eddy viscosity model produces the best agreement in the far field of an incompressible axisymmetric jet into calm surroundings. However, with the addition of a coflowing stream, the success of the Prandtl model deteriorates rapidly. In this case the recommended model is due to Ferri, Libby, and Zakkay.[3] For incompressible flow, Ferri's model reduces to Prandtl's model with a larger constant. Thus its predictions are not really improved but rather shifted. It is the purpose of this Note to demonstrate that the inclusion of a simple nondimensional term in the Prandtl model can greatly increase the success of the predictions in the far field of the incompressible coaxial jet.

Received April 19, 1972; revision received June 30, 1972. Prepared under Contract F33615-71-C-1463 with Technology Inc., Dayton, Ohio. Thanks are due to B. P. Quinn and P. M. Bevilaqua for their helpful comments and V. Grycz for typing the manuscript.

Index categories: Jets, Wakes, and Viscid-Inviscid Flow Interactions.

* Visiting Research Associate. Member AIAA.

Prandtl's model of the eddy viscosity ε, is

$$\varepsilon = kb(u_{max} - u_{min}) \quad (1)$$

where k is an empirical constant, b is proportional to the width of the mixing region and u_{max} and u_{min} are the maximum and minimum values of streamwise velocity at a given streamwise position.

The ability of the Prandtl model to predict the axis velocity decay of coaxial jets is shown in Fig. 1. The centerline velocity u, and the coflowing stream velocity u_e, are equivalent to u_{max} and u_{min}, respectively. The initial velocity of the jet is U, while the velocity ratio is $m \triangleq u_e/U$. The const in the Prandtl model is chosen as $k = 0.007$ in the region $u > 0.99U$ and $k = 0.011$ downstream of that region. The half width, b, is defined as the distance between positions on the velocity profile where the velocity is equal to half the centerline velocity.

A reasonable curve fit of the available data[1] is indicated by the heavy line decaying at x^{-1} (i.e., 45° slope). It may be seen that the agreement of the Prandtl model deteriorates as the velocity ratio m increases.

The Ferri model was formulated to account for mixing in variable density flows. When simplified to the incompressible case, it reduces to the Prandtl model with a higher const, $k = 0.025$. The results of this model are also shown in Fig. 1. The intersection point of the Prandtl model with the empirical curve has moved downstream somewhat, resulting in better agreement near that point. However, the far field decay slope still is not matched very well.

The original development of the Prandtl model assumed that the dimensions of the lumps of fluid which are transported across the jet are of the same order of magnitude as the width of the mean shear layer. That this assumption is affected by the addition of a coflowing stream may be seen in the limit where the coflowing stream velocity is equal to the jet velocity and the width of the mean shear layer is zero. The flow is, however, still turbulent and the lumps of fluid transported by the turbulence are of finite size. Thus as the coflowing stream velocity increases, the size of the fluid lumps can be substantially larger than the width of the shear layer. Increasing the dimensions of the fluid lumps is equivalent to increasing the magnitude of the effective viscosity. To incorporate this effect into the Prandtl model, the eddy viscosity is increased by a term inversely proportional to the local velocity difference between the jet and freestream $(U - u_e)/(u - u_e)$.

The denominator of the preceding term is equivalent to the $(u_{max} - u_{min})$ term in the Prandtl model, so if the Prandtl model were simply multiplied by the coflowing stream modification, the local velocity difference terms, $(u - u_e)$, would cancel out and the resulting eddy viscosity would be proportional to the half width alone. To avoid this, the modification factor is raised to a power characteristic of the flow situation, the ratio of the coflowing stream velocity to the jet velocity, m.

The use of the velocity ratio as the exponent in the modification term may be viewed as follows. The shortcoming of the Prandtl

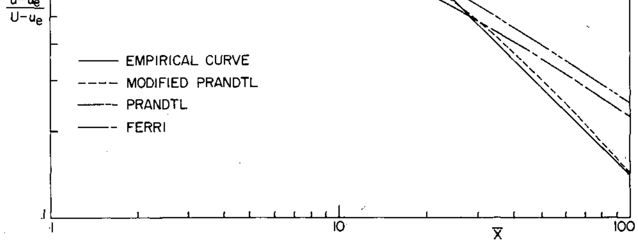

Fig. 1 Axial velocity decay; velocity ratio = 0.67.